

The “hard” problem of strong of interactions.

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Abstract

This is a write-up of a lecture at the level of a physics colloquium. There exists an idealized mathematical formulation of strong interactions which has no free parameters but is known to describe the real world quite accurately. Over the last three decades the problem has been managed with increasing success. An overview of some facts and a little fiction will be presented, but the question whether the problem can now be considered “easy” will be left unanswered.

1 Introduction

I am grateful for the honor to deliver this lecture. I received all my professional physics education in this department and the preparation of this talk made me aware of the influence this background has had on my research trajectory since graduation in 1979.

Joe (Yossef), my Ph. D. advisor, stood out among the Tel Aviv particle theorists of my time as a student: he spoke in complete sentences using proper grammar and enunciation. Also, unconventionally, he put content into every sentence. He did this naturally, without being overbearing. Here, I note two of his observations: (a) Physicists, as opposed to psychologists, know their variables. (b) A track record of “luck” should be kept, as representing features we do not understand.

As you will see, when it comes to strong interactions we do not always know the right variables, and, we need “luck” to find them.

2 The practice of physics

Using observation, guess rules. Then, apply the rules to real and imagined situations. Check that the results are consistent with other rules and agree with new or existing

observation. Eventually, some rules become established beyond reasonable doubt. These rules are never an ultimate truth, but must be well defined in an idealized framework.

A special situation may occur when well established rules are known to produce certain broad pervasive phenomena, but it is unknown how to show this from the rules directly, and/or make quantitative predictions about these phenomena. These are “hard” problems, and the theory of strong interactions presents us with such. Other examples, like high T_c superconductivity or turbulence exist outside particle physics.

Today, I’ll be talking about the “hard” problem of strong interactions, whose rules have been well established for 35 years by now.

3 Strong interactions

There are four major established force types in Nature and they all are associated with local symmetries: gravitational, weak, electromagnetic and strong. The best understood among them is the electromagnetic force. The least understood is the gravitational force, mainly because it is always attractive and this leads to instabilities. The LHC is hoped to clarify the nature of the weak force. The strong force is a spoiler, since anything one sees at the LHC is contaminated by strong force effects. Nowadays these effects are “managed”, but not cured. Two main properties of strong interactions need to be brought under control before achieving a cure: Chiral symmetry breaking and confinement.

A simplification of the strong interaction rules as they occur in Nature is usually employed: in it one ignores the “heavy” quarks and sets the masses of the “light” quarks, up, down and strange to zero. In addition one ignores all other forces, but the strong one. This produces a theory free of any parameters. Its rules must produce pure numbers quantifying a wide range of microscopic phenomena, among which chiral symmetry breaking and confinement are fundamental.

The rules are determined by the gauge symmetry group $SU(N_c)$ (of color) ($N_c = 3$), by the representation content of the quark fields ($\mathbb{N}_c, \overline{\mathbb{N}}_c$) and by their number (of flavors), ($N_f = 3$). The rules are most succinctly summarized by a Lagrangian, which is the integrand of S , the action:

$$\mathcal{S} = \frac{1}{4g^2} \int d^4x \text{Tr} F_{\mu\nu}^2(x) + \int d^4x \overline{\psi}(x) \gamma_\mu [\partial_\mu - iA_\mu(x)] \psi(x) \quad (1)$$

g is a device for expanding various predictions in the inverse of the logarithm of the overall scale of the process; g is not a true parameter. The integrals are over four dimensional space-time. The fields $\overline{\psi}$ and ψ represent Dirac spin 1/2 particles which carry two types of indices (quantum numbers) one flavor, running from 1 to N_f and the other color, running from 1 to N_c . The field A_μ represents spin 1 particles carrying a color index that takes values from 1 to $N_c^2 - 1$. These indices are repackaged by making A_μ a hermitian traceless $N_c \times N_c$ matrix. As Dirac spinors, $\overline{\psi}, \psi$ each have a spinorial fourfold valued index, acted on by the four 4×4 matrices γ_μ . These matrices are Clebsch-Gordan coefficients coupling the two spin 1/2 fields to a spin 1. The chromatic

field strength $F_{\mu\nu}$ is a traceless hermitian matrix, antisymmetric in μ, ν , defined by:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \quad (2)$$

\mathcal{S} is invariant under the change of variables $\partial_\mu - iA_\mu(x) \rightarrow h(x)[\partial_\mu - iA_\mu(x)]h^\dagger(x)$, with $h(x) \in SU(N_c)$, a property called “local gauge invariance”.

4 Hamiltonian and path integrals

The entire Lagrangian is fixed by the requirements of symmetry and minimalism. One keeps a minimal number of derivatives and minimal nonlinearity.

The Lagrangian can be used to construct a Hamiltonian slightly generalizing the prescription one learns in Classical Mechanics courses (I was taught CM by Joe) to situations where the standard definition of conjugate momenta does not produce independent variables. The classical Hamiltonian is then quantized.

A full definition of the theory requires an intermediate step at which the number of degrees of freedom is made finite. There are two types of infinities to control: the fact that x can run off to infinity (IR) and the fact that there is an infinite number of x -values in the smallest volume of space (UV). The control of the IR infinity is relatively simple. The control of the UV infinity is more subtle, requiring the regulating parameter to be taken to infinity while adjusting the operators in an elaborate, but well understood way.

Instead of the Hamiltonian framework one can use path integral quantization. Then the Lagrangian itself is used, and one defines an integral over all fields of the exponent of the action. Matters are helped by a trick, which takes time to the imaginary axis, and replaces the Lorentz group $O(3,1)$ by the euclidean group $O(4)$. One still needs to take care of IR and UV limits as before. The advantage of this formulation is that it offers a different way of understanding the gross features of the physics: one searches for “important” field configurations, which dominate the path integral for one type of observable or another. There is no such thing as a fermion field configuration. The integral over the fermion fields isn’t a generalization of an ordinary Riemann integral, unlike the integral over the A_μ fields. Rather, the integral over the fermions is defined by first keeping the A_μ ’s fixed, and then it gives:

$$\det[\gamma_\mu(\partial_\mu - iA_\mu)]^{N_f} \equiv \det[G_f^{-1}(A)]^{N_f} \quad (3)$$

Moments of fermion operators are defined in terms of the propagator $G_f(A)$.

Thus, as far as the fermions go only the inverse of an A_μ -dependent first order partial differential operator comes in:

$$G_f(A) = \frac{1}{\gamma_\mu(\partial_\mu - iA_\mu)} \quad (4)$$

This is a consequence of the fermions entering only bilinearly in the Lagrangian, a special feature of four spacetime dimensions and the requirement of minimality.

5 Large N_c

A further special simplification is made by setting $N_c g^2 = \lambda$ and taking $N_c \rightarrow \infty$ at fixed λ and N_f . Now λ will generate a scale. This large N_c limit (t' Hooft's [1]) is known to continue exhibiting spontaneous chiral symmetry breaking and confinement and in general to provide good approximations to a large number of observables. The number of A_μ -type degrees of freedom grows as N_c^2 , while that of ψ -type degrees of freedom grows only like N_c . The net result is that fermions are relegated to the role of spectators which means that the fermion determinant can be set to 1, and the “energy” cost of an A_μ -configuration is given by the classical action. (The “importance” of a configuration also depends on an “entropy”, in addition to the observable of concern.)

When the theory is regulated in the UV and IR, one discovers that the series in λ has a finite radius of convergence [2]. A similar expansion at finite N_c , in g , is only asymptotic. The Feynman diagrams representing the expansion in λ are planar while for an expansion in g there is no such restriction. The number of planar diagrams at fixed order only grows exponentially with the order, while the number of all diagrams at fixed order grows factorially and this is the reason for the distinct nature of the two series. The planar structure of diagrams allows one to draw several qualitative conclusions, if one assumes confinement. These conclusions are nicely born out by experiment. In short, a solution of the theory in the planar limit would take us a long way toward cracking the hard problem of strong interactions.

Increasing numerical evidence coming from lattice gauge theory indicates that at large distances the planar theory behaves like a theory of weakly self-interacting strings [3]. The strings do not interact with each other. For strings that touch, theoretically, one guesses that the closed string coupling constant behaves as $1/N_c^2$, while the open string constant goes as $1/N_c$. So far, this seems to work well for long distances. Still, it is doubtful that a useful string description exists for short distances.

Nevertheless, things look better in this respect nowadays [4]. It seems possible to find descriptions of $N_c = \infty$ strong interactions at large distances by noninteracting string-like states. These strings propagate on weakly curved higher dimensional spaces. Speculating that one can make a firm theory for these strings also on strongly curved spaces, one dreams of an “in principle” string description of planar Feynman diagrams within their regime of convergence.

6 Chiral symmetry

The four γ_μ matrices are all off diagonal in terms of 2×2 blocks: the matrix $\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ anticommutes with all γ_μ . Accordingly, the four components of $\bar{\psi}$ and ψ separately split into two two-components spinors each, $\bar{\psi}_{R,L}$ and $\psi_{R,L}$. Only $\bar{\psi}_R, \psi_R$ and $\bar{\psi}_L, \psi_L$ are directly coupled in the Lagrangian. In Hamiltonian language the RL split corresponds to a separation according to helicity among the quarks and the antiquarks. Helicity is a good quantum number only for massless particles. A fermion number conserving mass

term consists of $\bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L$. For zero mass, RL independent changes of basis in flavor space are a symmetry. Thus the Lagrangian has a $U_R(N_f) \times U_L(N_f)$ symmetry. The chiral symmetry group is $SU_R(N_f) \times SU_L(N_f)$ obtained from the original classical symmetry group after setting aside the two central $U(1)$'s.

In the functional integral formulation, chiral symmetry is special when compared with other global symmetries known in physics in that it holds even at fixed arbitrary A . So, configuration by configuration in the path integral, the symmetry holds. The more typical situation is that the symmetry holds only after averaging but is not preserved configuration by configuration.

Although the Hamiltonian commutes with the generators of chiral symmetry, the vacuum of the theory is not invariant under the entire group. This fact is supported by experiment. It turns out that the light quark masses are light enough that one can perturb in them around the massless case. This perturbation would produce the relatively light observed masses of pions and kaons if we assume that chiral symmetry is spontaneously broken at zero mass. The vacuum is preserved by the subgroup in which the action of the two factors in $SU_R(N_f) \times SU_L(N_f)$ is identical. The complementary subset, where the factors are conjugate of each other, is not an invariance of the vacuum. Each charge is a space integral of a local current by Noether's theorem. Integrals of the currents with slowly varying weights, when acting on the vacuum, create states distinct from the vacuum. When the weights are constant the energy of the excitation is zero because the charges commute with the Hamiltonian. For slowly varying weights, the energies are therefore low. These states are made out of massless scalar particles, the Goldstone bosons associated with the broken generators. These Goldstone bosons acquire small masses when we add $m\bar{\psi}\psi$ to the Hamiltonian with m small relatively to the generated scale. The masses square of these would-be Goldstone bosons are linear in the perturbation, that is the quark masses. Other mesons have larger masses that stay finite and large when the quark masses are taken to zero.

The operator $\bar{\psi}\psi$ is invariant under the subgroup preserved by the vacuum. Under the broken transformations the operator changes. Spontaneous breakdown in the vacuum reflects itself in the vacuum expectation value $\langle\bar{\psi}\psi\rangle$ being non-zero at $m = 0$. This quantity is usually referred to as the “condensate” and a fundamental symptom of spontaneous chiral symmetry breaking is the existence of a non-zero condensate. The condensate in the presence of the $m\bar{\psi}\psi$ term is a nonzero function of m . Consider first finite N_c , IR (L) and UV (a) cutoffs. We assume a UV lattice cutoff, so a denotes the lattice spacing. The IR cutoff comes in the form of a four torus of side L lattice spacings. Now the matrix G_f is a finite matrix of size $4L^4N_cN_f \times 4L^4N_cN_f$, depending on the gauge field and on the mass m . The structure in the flavor indices is a Kronecker delta function.

In the path integral formalism, using continuum notation, we have

$$\langle\bar{\psi}\psi\rangle \propto \int [dA_\mu] \det[\gamma_\mu(\partial_\mu - iA_\mu) + m]^{N_f} e^{-\frac{1}{4g^2} \int \text{Tr} F_{\mu\nu}^2} \text{tr} G_f(A; m; x, x) \quad (5)$$

If one takes the limit $m \rightarrow 0$ now the integrand vanishes, but if one first takes the limit $L \rightarrow \infty$ and only subsequently $m \rightarrow 0$ one gets a nonzero answer.

Let us sum over sites x and divide by L^4 . This does not change the answer obtained after performing the A integral. The summation over x means that the object $\text{tr}G_f(A; m; x, x)$ is replaced by $\text{Tr}G_f(A; m)$, that is, the trace is performed also on the x, y indices of the matrix G_f . This leads us to the eigenvalues of $G_f(A; m)$, which are just the inverses of the eigenvalues of $\gamma_\mu(\partial_\mu - iA_\mu) + m \equiv D(A) + m$. Since m enters additively, we only need the eigenvalues of the massless Dirac operator $iD(A)$. As L increases the latter become dense, and random with a distribution determined by the A -dependent weight. Since the trace is additive on the eigenvalues we only need the single eigenvalue density, $\rho_L(\mu) = \rho_L(-\mu)$. This density defines the average level spacing at μ . Extracting a factor of L^4 , we make the normalization of ρ finite in the $L \rightarrow \infty$ limit. We shall get a nonzero finite condensate if the normalized density of eigenvalues μ goes as a constant for $\mu \sim 0$ [5]:

$$\langle \bar{\psi}\psi \rangle \propto \int d\mu \rho(\mu) \left[\frac{1}{\mu + im} + \frac{1}{-\mu + im} \right] \quad (6)$$

One sees clearly how the integrand seems to vanish at $m = 0$, but its limit as $m \rightarrow 0$ is nonzero if $\rho(0)$ is nonzero.

Thus, a simple spectral property of the random matrix $D(A)$ around zero is equivalent to a nonzero condensate. Zero is a special point because, for every A , $\gamma_5 D(A) + D(A) \gamma_5 = 0$, which produces the pairing of $\pm\mu$ in the spectrum. $D^2(A)$, which commutes with γ_5 for every A , has a twofold degeneracy; in spinor space $D^2(A)$ is 2×2 block diagonal with one block being given by $W(A)W^\dagger(A)$ and the other by $W^\dagger(A)W(A)$ where $W(A)$ connects the RR components and $W^\dagger(A)$ connects the LL components. $W^\dagger W$ and WW^\dagger are isospectral. (I ignore zero modes and topology.)

The simplest random matrix probability distribution of a complex matrix W , otherwise unrestricted, provides exactly the type of spectral density required for spontaneous chiral symmetry breaking [6]. The conclusion is that about any sort of strong enough disorder induced by the fluctuations in A from configuration to configuration will produce a finite nonzero condensate. Only weak disorder will thin out the spectrum at 0 sufficiently to produce $\rho_\infty(0) = 0$ and a fully chirally invariant vacuum.

An even simpler picture emerges in the planar limit. First, the determinant factor can be ignored. Second, one does not need to take L to infinity. A moderate L is enough and produces the same $N_c \rightarrow \infty$ limit as an infinite L would [7]. Third, at finite L and a , as $N_c \rightarrow \infty$, a finite fraction of the entries of $W(A)$ remain random numbers.

It has taken about 30 years for this understanding of chiral symmetry breaking to develop. Starting from noticing the connection between $\rho(0)$ and $\langle \bar{\psi}\psi \rangle$ at a formal level, it has been necessary to first UV regulate in a way which preserved chiral symmetry to assure that the realization of chiral symmetry indeed is a purely IR phenomenon [8], unrelated to the UV. Next, it took some time to realize the relation to random matrix theory, and to show that the eigenvalue density, and the ordered eigenvalues themselves, can be renormalized in a meaningful way. Further, the simplification at infinite N_c was understood only relatively recently [9].

7 Confinement

Mesons are made out of a quark and an antiquark of possibly different flavors and “extra” stuff. Mesons can have a high spin, J , and mass M . Thinking about a blob of radius R one expects $J \propto R^2$. Experimentally one finds $J \propto M^2$ and hence $M \propto R$. This indicates that the two quarks are in a linearly rising potential for R large enough. A simple explanation for this is that the quark pair is connected by a straight tube of constant chromatic energy, a flux tube of uniform structure along its length.

Whether the quarks are moving and have an angular momentum or stationary, the flux tube should be the same – so one guesses. This leads to a criterion for confinement [10]:

$$\text{Tr}W(\mathcal{C}) \equiv \frac{1}{N_c} \text{Tr} \langle e^{i \oint_{\mathcal{C}} A \cdot dx} \rangle \sim e^{-\sigma \mathcal{A}} \quad (7)$$

Here \mathcal{C} is a closed loop of rectangular shape $R \times T$, $\mathcal{A} = RT$ and one takes $T \rightarrow \infty$ at fixed large R . The static potential is $V(R) \sim \sigma R$, exhibiting confinement. This is a particular geometry realizing the more general statement that the area law holds for any simple enough loop with a unique minimal area \mathcal{A} , asymptotically as the loop is uniformly blown up.

For the rectangular geometry, one can think of the operator as describing the amplitude, in Euclidean space, of a state consisting of quark of mass $M = \infty$ fixed at a point and its antiquark situated at distance R away. The amplitude would have been $e^{itE(R)}$ in real time, becoming $e^{-TV(R)}$ after rotating the time to the imaginary axis and dropping the infinite factor e^{-2MT} .

If the gauge theory were abelian, the line integral in the exponent would represent the magnetic flux through any area spanning the curve. Clearly, to get such a small average, the flux must fluctuate strongly. A simple way to understand the area law would be to assume that the flux is additively made out of small contributions, each associated with small patches tessellating the minimal area spanning the curve and fluctuating independently of each other, in accordance with identical laws.

In a nonabelian theory this picture needs a reformulation: there is no additive non-abelian flux, the role of a, specifically *minimal* spanning area is difficult to understand due to the nonlinearity of the nonabelian equations, and the trace operation should have a role. Moreover, since the force carriers in the nonabelian case carry charge, one must understand why the sources are not screened spontaneously. The screening effect is impossible to argue away dynamically, but the argument has a limitation. The charge of the force carriers is like that of a quark antiquark pair with free color indices, with the trace subtracted. The group $SU(N_c)$ has elements corresponding to a unit matrix times a phase $e^{\frac{2\pi i k}{N_c}}$, $k = 0, 1, \dots, N_c - 1$; these “center” elements commute with the entire group and make up an abelian $Z(N_c)$ group. Quarks change when acted on by any $k \neq 0$ element by a phase and antiquarks by the opposite phase, leaving the gluons, the force carriers, unchanged. So, any external charges with “ N -ality” $k \neq 0$ cannot be screened. There should be $N_c - 1$ distinct string tensions, one for each k , σ_k .

So, our abelian mechanism might work, if we keep it restricted to the center $Z(N_c)$.

We still need to detach the concept of random flux contributions from the minimal area. The idea is to associate the flux contributions to other loops, which carry flux and link the loop of our observable. Since loops only link in 3D, we need to adopt a Hamiltonian picture and abandon, for the time being, the path integral. We imagine the Hilbert space as made out of functionals of the space components of A_μ . As the gauge invariance is local, one needs to define the representation content of each point in space. One places constraints on the type of wave functionals that are acceptable: they must obey Gauss's law, making them singlets. We now view W as an operator, acting by multiplication on the wave functionals. W is gauge invariant, so it acts within the Hilbert space. The confinement criterion is a statement about the vacuum expectation value of the operator W . We now define another operator associated with curves, $B_k(\mathcal{C})$, which obeys [11]: $B_k(\mathcal{C})W_{k'}(\mathcal{C}') = e^{\frac{2\pi i l k k'}{N_c}} W_{k'}(\mathcal{C}') B_k(\mathcal{C})$.

Here k, k' denote possible N -alities and l is the number of times the loops link. B creates a tube of $Z(N_c)$ magnetic flux along its curve producing a small patch of flux on any area spanning the curve of W at the point where the B -curve pierces it. Piercing has to happen if the curve of B links that of W . Two loops can link and still be distant from each other and this makes the commutation relation a nontrivial constraint in a theory describable by a local Lagrangian. The commutation relation shows that the W, B are exponential versions of canonically conjugate pairs.

If the operator B condenses in the vacuum, one can imagine these small patches to be independent of each other and produce confinement. Alternatively, it may be that W condenses, in which case B would obey an area law. The operator that condenses, itself, cannot obey an area law if all particles become massive, as the energy cost would now be concentrated at the perimeter. By “condenses” one means, roughly, that the vacuum state is a coherent state relative to that operator. It is also possible to not condense neither W nor B , in which case both operators obey perimeter laws, but then massless particles have to be present in order to realize the commutation rule. Is it possible to condense both, in which case both operators have area laws.

The eigenvalues of the formally unitary matrix $e^{i \oint A \cdot dx}$ are also gauge invariant and the action of B moves them all cyclically round the circle in steps of $\frac{2\pi}{N_c}$. For small loops we know that the nonlinearities are weak and therefore the eigenvalues will concentrate near unity. Thus, if B condenses, the patches of flux should have a typical physical scale, so that the eigenvalues associated with the small loop can be nonuniform round the unit circle.

At infinite N_c the eigenvalues freeze. The reason is that there are only $N_c - 1$ of them while the probability distribution governing them has an exponent that must go as N_c^2 for large N_c . In addition, for large N_c the eigenvalues form a continuum, just as before when we were discussing the eigenvalues of the Dirac operator. For a small enough loop the support of the eigenvalue distribution does not even reach round the unit circle. For a large loop we need an almost uniform distribution, caused by the fluctuations of the patches of flux associated with the condensation of B . So, we learn that at infinite N_c the world of small loops is nonanalytically different from that of large loops. Again, the precise mechanism behind this transition as one watches a dilating loop is not important

except that it induces enough randomness among the eigenvalues. The condensation of B is an agent causing it, but many other effects could also cause it. After all, even with a perimeter law the eigenvalues still would have to spread out over the entire unit circle. B condensation enters only for large loops, when one needs to explain the fact that $\sigma_k \neq 0$ for $k \neq 0$. The transition between a gapped distribution of eigenvalues to an ungapped one (on the unit circle) is of a generic type, likely universal.

To understand how the strong interaction forces go from being weaker than electromagnetic forces at short distances to being confining at long distances, we need to gain control of this transition at infinite N_c . It is replaced by a narrow crossover at finite N_c [12].

One problem at $N_c = 3$ is that the loop-operator framework is defined in Hamiltonian language, while our basic validating technique is to search for important configurations in the Euclidean path integral. There has been ongoing research on this for two decades. In spite of strong support of the picture I reviewed, no overwhelming evidence has been yet found [13].

The understanding of the crossover at large N_c is a relatively new activity. It's main objective is to lay the ground for a possible future quantitative approach to QCD at all scales, which circumvents the problem of identifying the source of confinement by using the paradigm of matched effective theories to connect the field theory of short distances to a string theory description of large distance effects.

8 Summary

In summary, the phenomena of spontaneous chiral symmetry breaking and confinement look nowadays less mysterious, but, I for one, would not agree that the “hard” problem of strong interactions is already under control.

We are still looking for the right variables, and we could use some luck in our search.

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